

Testing Time Series Linearity: Traditional and Bootstrap Methods

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Abstract

We review the notion of time series linearity and describe recent advances in linearity and Gaussianity testing via data resampling methodologies. Many advances have been made since the first published tests of linearity and Gaussianity by Subba Rao and Gabr in 1980, including several resampling-based proposals. This chapter is intended to be instructive in explaining and motivating linearity testing. Recent results on the validity of the AR-sieve bootstrap for linearity testing are reviewed. In addition, a subsampling-based linearity and Gaussianity test is proposed where asymptotic consistency of the testing procedure is justified.

Keywords: AR-sieve, asymptotic consistency, bootstrap, gaussianity testing, linearity testing, literature review, subsampling, time series.

1. Introduction

Ever since the fundamental recognition of the potential role of the computer in modern statistics (Efron, 1979a,b), the bootstrap and other resampling methods have been extensively developed for inference in independent data settings; see, e.g., the works done by Davison and Hinkley (1997), Efron and Tibshirani (1993), Hall (1997), Shao and Tu (1995). Such methods are even more important in the context of dependent data where the distribution theory for estimators and test statistics may be difficult to obtain even asymptotically.

In the time series context, different resampling and subsampling methods have been proposed, and are currently receiving the attention of the statistical community. Reviews of the impact of bootstrap methods on time series analysis may be found in books (Lahiri, 2003; Politis et al., 1999), papers (Bühlmann, 2002; Politis, 2003), and the review by J.-P. Kreiss and S. N. Lahiri in this volume of the Handbook.

In the paper at hand, we revisit the problem of assessing whether a given time series is linear versus nonlinear, or Gaussian versus non-Gaussian. In practice, a Gaussian classification would indicate an Autoregressive Moving Average (ARMA) model with Gaussian innovations is appropriate, whereas a linear classification would indicate that an ARMA model with independent but possibly non-Gaussian innovations can still be considered. However, the rejection of linearity typically requires the practitioner to carefully select an appropriate nonlinear model for the underlying time series, or even to proceed in a model-free, nonparametric manner.

We review the traditional linearity and Gaussianity tests that are based on the normalized bispectrum. The critical regions of these tests have been traditionally determined via asymptotic methods. As an alternative, we describe how these critical regions can be determined via resampling (e.g., the AR-sieve bootstrap) and/or subsampling. One of the advantages of subsampling methodology is the generality under which it is valid. There are a number of examples where subsampling yields consistent estimation but the bootstrap fails (Politis et al., 1999). Although subsampling is more widely applicable, it is noted that when the bootstrap is indeed valid it may possess second-order asymptotic properties (Hall, 1997) giving the bootstrap an advantage.

The literature on linearity and Gaussianity tests is reviewed in the next section. The concept of time series linearity is thoroughly described in Section 3. Sections 4 and 5 focus on the AR-sieve bootstrap and subsampling tests, respectively.

2. A brief survey of linearity and Gaussianity tests

Several parametric and semiparametric tests of linearity designed with a specific nonlinear model as an alternative hypothesis have been proposed, including the works of An et al. (2000), Ashley and Patterson (2009), Chan (1990), Chan and Tong (1986, 1990), Hansen (1999), Harvey and Leybourne (2007), Keenan (1985), Luukkonen et al. (1988), Petrucci (1990), Petrucci and Davies (1986), Terasvirta (1994), Terasvirta et al. (1993) and, Tsay (1986). Some tests have model-based assumptions on the null hypothesis (e.g., assuming the null to be $AR(p)$ where p may or may not be assumed known) and some tests induce model-based assumptions on the alternative hypothesis (e.g., assuming the specific GARCH nonlinear alternative hypothesis). Such model-based assumptions may help to increase the power of the various tests, but only when the respective assumptions are satisfied.

Many nonparametric or model-free tests, including the first published linearity test done by Subba Rao and Gabr (1980), are based on nonparametric estimates of the normalized bispectrum, and thus involve much less restrictive assumptions under the null and alternative hypothesis; the normalized bispectrum will be defined and discussed in Section 3. Further bispectrum-based tests include the tests done by Ashley et al. (1986), Birkelund and Hanssen (2009), Brockett et al. (1988), Hinich (1982), Jahan and Harvill

(2008), Subba Rao and Gabr (1984), and Yuan (2000). Tests based on the normalized bispectrum are frequently used in practice when data are available in abundance, for example, when analyzing financial time series; see, e.g., the works done by Abhyankar et al. (1995, 1997), Hinich and Patterson (1985, 1989), and Hsieh (1989). Note that there are other nonparametric or model-free tests of linearity that are not based on the normalized bispectrum; see, e.g., the works done by Hong-Zhi and Bing (1991), Terdik and Math (1998), and Theiler et al. (1992). An overview of some of these tests is provided in the works of Corduas (1994). In this volume, Giannerini (2011) provides an overview of several linearity testing approaches.

Because of the nonparametric nature of the bispectrum-based tests, their critical regions have traditionally been determined via asymptotic approximations. However, considerably large sample sizes can be necessary in order to accurately estimate the two-dimensional bispectral density. As such, a number of resampling-based methods have been proposed in the recent literature to overcome this limitation in a finite sample size setting.

There are many published reports, especially in recent years, that utilize some form of resampled data in linearity testing (Berg et al., 2010; Birkelund and Hanssen, 2009; Hinich et al., 2005; Hjellvik and Tjøstheim, 1995; Kugiumtzis, 2008). Many of these methods involve bootstrapping residuals obtained from fitting a parametric model which is equivalent to resampling the data obtained after a prewhitening step that has removed (to large extent) the presence of autocorrelation. If the prewhitening is performed by fitting an autoregressive [AR(p)] model to the data, then typically practitioners would choose the order p in a data-dependent manner, say by minimizing an information criterion such as AIC, BIC, etc. In practice, it is extremely rare that a finite-order AR(p) would explain the data perfectly; more often than not, the practitioner would use an order p that would be an increasing function of the sample size n , thereby creating an approximating *sieve* of AR models. This is the essence of the AR-sieve bootstrap that is reviewed in detail in the chapter by J.-P. Kreiss and S. N. Lahiri in this volume of the Handbook; the application of the AR-sieve bootstrap to linearity testing is discussed in Section 4.

Another popular approach for linearity testing is the *surrogate data* approach of Theiler et al. (1992). The idea of the surrogate data¹ method is to apply the bootstrap on the phases of the Discrete Fourier Transform (DFT) of the data while keeping the magnitudes of the DFT unchanged. With an inverse DFT, bootstrap pseudo-series can then be created. It is immediate that these pseudo-series have identical second-order structure as the original series, since the second-order structure is coded in the periodogram which remains unchanged in this process.

Alternative uses of the bootstrap in the literature of linearity and Gaussianity testing include a phase scrambling bootstrap (Barnett and Wolff, 2005), the use of bootstrapped residuals to obtain the correct false alarm rate (Hinich et al., 2005; Birkelund and Hanssen, 2009), and the Time Frequency Toggle (TFT)-bootstrap (Kirch and Politis, 2011). The TFT-bootstrap can actually be seen as a generalization of the

¹ In this chapter, we reserve the term *surrogate data* for the method of Theiler et al. (1992); however, the reader should be warned that other authors use the term as a generic way of referring to bootstrap data including even the AR-sieve bootstrap (Hinich et al., 2005; Theiler and Prichard, 1997).

surrogate data method since it involves resampling of both the phases and the magnitudes of the Fourier coefficients. Several surrogate and bootstrap tests for linearity in time series were compared by Kugiumtzis (2008). Finally, a different test has recently been proposed that combines an entropy measure of (non)linearity with bootstrap critical regions (Giannerini et al., 2011).

In this chapter, we chose to highlight two resampling-based tests for time series linearity and Gaussianity. The first is the aforementioned AR-sieve method that bootstraps the residuals obtained from an appropriate $AR(p)$ fit. The AR-sieve methodology has been popular for quite some time but its validity for testing Gaussianity or linearity has only recently been proven (Berg et al., 2010); it is discussed in Section 4. In Section 5, we also describe in detail a novel subsampling-based approach to Gaussianity and linearity testing. The next section defines and discusses the notion of linearity in time series.

3. Linear and nonlinear time series

Consider data X_1, \dots, X_n arising from a strictly stationary time series $\{X_t\}$ that – for ease of notation – is assumed to have mean zero.² The most basic tool for quantifying the inherent strength of dependence is given by the autocovariance function $\gamma(k) = EX_t X_{t+k}$ and the corresponding Fourier series $f(w) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-iwk}$; the latter function is termed the *spectral density*, and is well defined (and continuous) when $\sum_k |\gamma(k)| < \infty$. We can also define the autocorrelation function (ACF) as $\rho(k) = \gamma(k)/\gamma(0)$. If $\rho(k) = 0$ for all $k > 0$, then the series $\{X_t\}$ is said to be a *white noise*, i.e., an uncorrelated sequence; the reason for the term ‘white’ is the constancy of the associated spectral density function.

The function $\gamma(k)$ represents the second-order moments of the time series $\{X_t\}$; more technically, it represents the second-order *cumulants* (Brillinger, 2001; Rosenblatt, 1985). The third-order cumulants are encapsulated by the function $\Gamma(j, k) = EX_t X_{t+j} X_{t+k}$ and the resulting two-dimensional Fourier series

$$f(w_1, w_2) = (2\pi)^{-2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \Gamma(j, k) e^{-iw_1 j - iw_2 k}$$

is termed the *bispectral density*. For reasons to be apparent soon, we also define the *normalized bispectrum* as

$$K(w_1, w_2) = \frac{|f(w_1, w_2)|^2}{f(w_1) f(w_2) f(w_1 + w_2)}.$$

We can similarly define the cumulants of higher order whose corresponding multi-dimensional Fourier series are termed higher order spectral densities or *polyspectra*; see Section 5 for details. The set of cumulant functions of *all* orders, or equivalently the set of *all* higher order spectral density functions, is a complete description

² Centering the data at their sample mean (instead of the true mean) is perfectly acceptable for the subsequent discussion as the resulting error is negligible.

of the dependence structure of the general time series $\{X_t\}$. Of course, working with an infinity of functions is intractable; a welcome shortcut is offered by the notion of linearity.

A time series $\{X_t\}$ is called *linear* if it satisfies an equation of the type:

$$X_t = \sum_{k=-\infty}^{\infty} \beta_k Z_{t-k}, \quad (1)$$

where the coefficients β_k are (at least) square-summable, and the series $\{Z_t\}$ is independent, identically distributed (i.i.d.) with mean zero and variance $\sigma^2 > 0$. To avoid the confounding of the β 's with the scale parameter σ , it is helpful to assume that $\beta_0 = 1$.

A linear time series $\{X_t\}$ is called *causal* if $\beta_k = 0$ for $k < 0$, i.e., if

$$X_t = \sum_{k=0}^{\infty} \beta_k Z_{t-k}. \quad (2)$$

Equation (2) should not be confused with the Wold decomposition that *all* purely nondeterministic time series possess (Hannan and Deistler, 1988). In the Wold decomposition, the “error” series $\{Z_t\}$ is only assumed to be a white noise and not i.i.d.; the latter assumption is much stronger. The causality assumption has been used successfully in the context of nonlinear time series as well; see, e.g., the works of [Gourieroux and Jasiak \(2005\)](#) and [Wu \(2005\)](#).

Linear time series are easy objects to work with since their dependence structure is perfectly captured by the sequence of coefficients $\{\beta_k\}$. To elaborate, if $\{X_t\}$ satisfies Eq. (1), then its autocovariance and spectral density functions are given by $\gamma(k) = \sigma^2 \sum_{s=-\infty}^{\infty} \beta_s \beta_{s+k}$ and $f(w) = (2\pi)^{-1} \sigma^2 |\beta(w)|^2$, respectively; here $\beta(w)$ is the Fourier series of the β_k coefficients, i.e., $\beta(w) = \sum_{k=-\infty}^{\infty} \beta_k e^{iwk}$. In addition, the bispectral density is simply given by

$$f(w_1, w_2) = (2\pi)^{-2} \mu_3 \beta(-w_1) \beta(-w_2) \beta(w_1 + w_2), \quad (3)$$

where $\mu_3 = EZ_t^3$ is the third moment of the errors. Similarly, all higher order spectra can be calculated in terms of $\beta(w)$.

It is now apparent that the normalized bispectrum $K(w_1, w_2)$ satisfies:

$$K(w_1, w_2) = \frac{|f(w_1, w_2)|^2}{f(w_1)f(w_2)f(w_1 + w_2)} \stackrel{\text{linearity}}{=} \frac{(\mu_3)^2}{(2\pi)^2 \sigma^6} \stackrel{\text{Gaussianity}}{=} 0.$$

As indicated by the right-hand side of the above equation, when the time series is in fact linear, the normalized bispectrum will be constant. Furthermore, if the time series is Gaussian (and therefore also linear), the normalized bispectrum will be constantly equal to zero. These two observations form the basis for a host of test of linearity and/or Gaussianity starting with the original paper of [Subba Rao and Gabr \(1980\)](#). Note, however, that although linearity implies the normalized bispectrum is constant, the converse is not necessarily true. Thus there is the implicit, though presumably unlikely, limitation in producing a falsely negative result in the presence of certain nonlinear or non-Gaussian processes.

A prime example of a linear time series is given by the autoregressive (AR) family in which the time series $\{X_t\}$ satisfies a linear relationship with respect to its own lagged values, namely

$$X_t = \sum_{k=1}^p \theta_k X_{t-k} + Z_t \quad (4)$$

with the error process $\{Z_t\}$ being i.i.d. $(0, \sigma^2)$ as in Eq. (1). AR modeling lends itself ideally to the problem of predicting future values of the time series; this is particularly true if the AR model is causal. Causality of an AR model is ensured if all roots of the characteristic polynomial $1 - \sum_{k=1}^p \theta_k z^k$ have modulus greater than one; see, e.g., the works of [Brockwell and Davis \(2009\)](#).

For example, let \hat{X}_{n+1} denote the predictor of X_{n+1} on the basis of the observed data X_1, \dots, X_n . It is well known ([Billingsley, 1995](#)) that the optimal predictor with respect to Mean Squared Error is given by the conditional expectation, i.e., $\hat{X}_{n+1} = E(X_{n+1} | X_1, \dots, X_n)$. Thus, $\hat{X}_{n+1} = g_n(X_1, \dots, X_n)$ where $g_n(\cdot)$ is a (generally nonlinear) function of the data X_1, \dots, X_n . In the case of a *causal* AR model, however, it is easy to show that the function $g_n(\cdot)$ is actually *linear*, and that $\hat{X}_{n+1} = \sum_{k=1}^p \theta_k X_{n+1-k}$. Note also the property of “finite memory” in that the prediction function $g_n(\cdot)$ is only sensitive to its last p arguments. Although the finite memory property is specific to finite-order causal AR (and Markov) models, the linearity of the optimal prediction function $g_n(\cdot)$ is a property shared by all *causal* linear time series satisfying Eq. (2); this broad class includes all causal and invertible, i.e., “minimum-phase” ([Rosenblatt, 2000](#)), ARMA models with i.i.d. innovations.

However, the property of linearity of the optimal prediction function $g_n(\cdot)$ is shared by a larger class of processes. To define this class, consider a weaker form of (2) that amounts to relaxing the i.i.d. assumption on the errors to the assumption of a martingale difference, i.e., to assume that

$$X_t = \sum_{i=0}^{\infty} \beta_i v_{t-i}, \quad (5)$$

where $\{v_t\}$ is a stationary martingale difference adapted to \mathcal{F}_t , the σ -field generated by $\{X_s, s \leq t\}$, i.e., that

$$E[v_t | \mathcal{F}_{t-1}] = 0 \quad \text{and} \quad E[v_t^2 | \mathcal{F}_{t-1}] = 1 \quad \text{for all } t. \quad (6)$$

As in the study by [Kokoszka and Politis \(2011\)](#), we will use the term *weakly linear* for a time series $\{X_t\}$ that satisfies Eqs. (5) and (6). As it turns out, the linearity of the optimal prediction function $g_n(\cdot)$ is shared by *all* members of the family of weakly linear time series;³ see, e.g., Theorem 1.4.2 of [Hannan and Deistler \(1988\)](#).

³ Nonetheless, the class of time series for which the best predictor is linear is larger than the family of weakly linear series. A prime example of a nonweakly linear time series that actually admits a linear optimal predictor can be given by a series of *squared* financial returns, i.e., when the series $\{X_t\}$ satisfies $X_t = r_t^2$ for all t , and $\{r_t\}$ is modeled by an ARCH/GARCH model; see the works done by [Kokoszka and Politis \(2011\)](#) for details.

The family of Gaussian sequences is an interesting subset of the class of linear time series. Gaussian series occur when the series $\{Z_t\}$ of Eq. (1) is i.i.d. $N(0, 1)$, and they too exhibit the useful linearity of the optimal prediction function $g_n(\cdot)$. To see this, recall that the conditional expectation $E(X_{n+1}|X_1, \dots, X_n)$ turns out to be a linear function of X_1, \dots, X_n when the variables X_1, \dots, X_{n+1} are jointly normal (Brockwell and Davis, 2009).

Furthermore, in the Gaussian case all spectra of order higher than two are identically zero. It follows that all dependence information is concentrated in the spectral density $f(w)$. Thus, the investigation of the dependence structure of a Gaussian series can focus on the simple study of second-order properties, namely the ACF $\rho(k)$ and/or the spectral density $f(w)$. For example, an uncorrelated Gaussian series, i.e., one satisfying $\rho(k) = 0$ for all k , necessarily consists of independent random variables. Note that to check/test whether an estimated ACF, denoted by $\hat{\rho}(k)$, is significantly different from zero, the Bartlett confidence limits are typically used. Bartlett's formula, however, is only valid for linear or weakly linear time series (Francq and Zakoian, 2009; Hannan and Deistler, 1988; Romano and Thombs, 1996). In the (potentially) nonlinear case, even testing the simple null hypothesis $\rho(1) = 0$ becomes highly nontrivial, and is greatly facilitated by a computer intensive methods such as resampling or subsampling (Politis, 2003, Romano and Thombs, 1996).

4. AR-sieve bootstrap tests of linearity

The popular AR-sieve bootstrap method has also been recently shown to be an effective and robust method for Gaussianity and linearity testing. The following gives the general AR-sieve bootstrap algorithm, including separate procedures for Gaussianity and linearity testing as well as third possibility that sits between Gaussianity and linearity – a linear process with symmetric (though possibly non-Gaussian) innovations. The proof of asymptotic consistency of this procedure – under both the null and the alternative hypotheses – can be found in the study by Berg et al. (2010) along with simulations demonstrating its finite sample effectiveness.

AR-sieve bootstrap Algorithm

Step 0: According to some criterion (AIC, BIC, etc.), choose the order p of the AR(p) model to fit to the data $\mathbf{X} = \{X_1, X_2, \dots, X_n\}$.

Step 1: Fit an AR(p) model to $\{X_t\}$ with estimated coefficients $\hat{\theta}_p = (\hat{\theta}_{1,p}, \hat{\theta}_{2,p}, \dots, \hat{\theta}_{p,p})$; i.e., $\hat{\theta}_p$ is an estimator for θ_p where

$$\theta_p = (\theta_{1,p}, \theta_{2,p}, \dots, \theta_{p,p}) = \arg \min_{(c_1, \dots, c_p)} \mathbb{E} \left[\left(X_t - \sum_{j=1}^p c_j X_{t-j} \right)^2 \right].$$

Step 2: Let $\mathbf{X}^* = \{X_1^*, X_2^*, \dots, X_n^*\}$ be a series of n pseudo-observations generated by

$$X_t^* = \sum_{j=1}^p \hat{\theta}_{j,p} X_{t-j}^* + u_t^* \quad (t = -b, -b+1, \dots, 0, 1, \dots, n) \quad (7)$$

where $X_t^* := 0$ for $t < -b$; the positive number b denotes the so-called ‘burn-in’ period to ensure (approximate) stationarity of the bootstrap series.

In (7), the u_t^* 's are iid random variables having mean zero and distribution function F_n which is selected based on the purpose of the analysis. One of three distribution functions can be selected depending on the null hypothesis under consideration:

Linear null ($H_0^{(1)}$): If the null hypothesis states the time series is linear, then set $F_n = F_n^{(1)}$ to be the empirical distribution function of the centered residuals $\hat{u}_t - \bar{u}_n$, where

$$\hat{u}_t = X_t - \sum_{j=1}^p \hat{\theta}_{j,p} X_{t-j} \quad (t = p, p+1, \dots, n)$$

and

$$\bar{u}_n = \frac{1}{n-p} \sum_{t=p+1}^n \hat{u}_t.$$

Linear symmetric null ($H_0^{(2)}$): If the null hypothesis states the time series is linear with a symmetric distribution of errors, then set $F_n = F_n^{(2)}$ to be a symmetrized version of $F_n^{(1)}$ obtained by setting $u_t^* = S_t u_t^+$ with $S_t \stackrel{\text{iid}}{\sim} \text{unif}\{-1, 1\}$ (the discrete uniform distribution on -1 and 1) and $u_t^+ \sim F_n^{(1)}$.

Gaussian null ($H_0^{(3)}$): If the null hypothesis states the time series is linear with Gaussian errors, then set $F_n = F_n^{(3)} = N(0, \hat{\sigma}_p^2)$, where

$$\hat{\sigma}_p^2 = \frac{1}{n-p} \sum_{t=p+1}^n (\hat{u}_t - \bar{u}_n)^2.$$

Step 3: Compute $T(X^*)$ from the bootstrap series X^* where $T(\cdot)$ is the chosen statistic for the null hypothesis of interest. In the next section, examples of such statistics are provided for testing Gaussianity and linearity.

Repeat: Steps 2 and 3 are repeated a large number (say B) of times. The empirical distribution of the B bootstrap pseudo-statistics can then be used to approximate the true distribution of $T(X)$ under the null hypothesis thus making the test feasible.

For example, consider the aforementioned tests based on nonparametric estimates of the normalized bispectrum. In testing for linearity, the normalized bispectral estimator is evaluated over a grid of points and the variability of the estimates are quantified by the interquartile range. If the time series is in fact nonlinear, then the normalized bispectrum should exhibit great variability yielding an interquartile range larger than what would have been expected under linearity. Therefore, linearity is rejected for large values of the estimated interquartile range; see [Section 5.2](#) for more details. Traditionally, the threshold of such a test has been determined from the asymptotic distribution

of the test statistic under the null; the AR-sieve bootstrap offers us a nonasymptotic alternative critical value – see the works done by [Berg et al., 2010](#) for details.

In closing, note that a new bootstrap method for time series, the Linear Process Bootstrap (LPB), has been recently introduced ([McMurry and Politis, 2010](#)). The LPB generates linear time series in the bootstrap world whether the true model is linear or not, i.e., under the null of linearity but also under the alternative. As in the AR-sieve bootstrap case, this property makes the LPB bootstrap a promising alternative in connection with bootstrapping the test of linearity.

5. Subsampling tests of linearity

The general subsampling methodology for time series approximates the distribution of a statistic by evaluating the statistic on subsampled blocks or contiguous subsets of the original time series. As with any resampling procedure, there are certain assumptions required on the data and the statistic to guarantee convergence; however, the assumptions needed to achieve consistency of subsampling are generally weaker or easier to verify than the assumptions required for bootstrap procedures ([Politis et al., 1999](#)).

To fix ideas, we consider in detail two statistics: a linearity test statistic, t_n^L , and a Gaussianity test statistic, t_n^G . These test statistics are derived from estimates of the normalized bispectrum, and they are based on the statistics originally proposed by [Hinich \(1982\)](#). Whereas Hinich utilized asymptotic theory to determine the distribution of the statistics under their respective null hypotheses, the approach described here uses subsampling to approximate the distributions of the statistics.

The test statistics t_n^L and t_n^G are described and the asymptotic conditions needed to justify the subsampling tests based on these statistics are provided. These test statistics are based on estimates of the spectral density and the bispectrum. Therefore, we first present some theory for polyspectral inference followed by the bispectrum-based method of linearity and Gaussianity testing.

5.1. Kernel-based polyspectral estimation

The assumption of s th-order stationarity is required to define the s th-order polyspectrum. It requires that all moments of order $m \leq s$ to exist and be lag-invariant, i.e.,

$$E[X_{\tau_1} X_{\tau_2} \cdots X_{\tau_m}] = E[X_{\tau_1+t} X_{\tau_2+t} \cdots X_{\tau_m+t}]$$

for any set of integers τ_1, \dots, τ_m and t . This assumption lies between the weaker assumption of covariance-stationarity (same as second-order stationarity and wide sense stationarity) and the stronger assumption of strict stationarity (also known as strong stationarity).

Let X_1, X_2, \dots, X_n be a realization of an s th-order stationary time series with (possibly nonzero) mean μ . The s th-order joint cumulant is defined as

$$C(\tau_1, \dots, \tau_{s-1}) = \sum_{(v_1, \dots, v_p)} (-1)^{p-1} (p-1) \mu_{v_1} \cdots \mu_{v_p}, \quad (8)$$

where the sum is over all partitions (ν_1, \dots, ν_p) of $\{0, \dots, \tau_{s-1}\}$ and $\mu_{\nu_j} = E \left[\prod_{\tau_i \in \nu_j} X_{\tau_i} \right]$; refer to the works done by [Jammalamadaka et al. \(2006\)](#) for another expression of the joint cumulant. The s th-order spectral density is defined as

$$f(\boldsymbol{\omega}) = \frac{1}{(2\pi)^{s-1}} \sum_{\boldsymbol{\tau} \in \mathbb{Z}^{s-1}} C(\boldsymbol{\tau}) e^{-i\boldsymbol{\tau} \cdot \boldsymbol{\omega}}, \quad (9)$$

where the bold-face notation $\boldsymbol{\omega}$ denotes an $(s-1)$ -dimensional, vector argument, i.e., $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{s-1})$. We adopt the usual assumption on $C(\boldsymbol{\tau})$ that it be absolutely summable, thus guaranteeing the existence and continuity of the spectral density. A natural estimator of $C(\boldsymbol{\tau})$ is given by

$$\widehat{C}(\tau_1, \dots, \tau_{s-1}) = \sum_{(\nu_1, \dots, \nu_p)} (-1)^{p-1} (p-1)! \hat{\mu}_{\nu_1} \cdots \hat{\mu}_{\nu_p}, \quad (10)$$

where the sum is overall partitions of (ν_1, \dots, ν_p) of $\{0, \dots, \tau_{s-1}\}$ and

$$\hat{\mu}_{\nu_j} = \frac{1}{n - \max(\nu_j) + \min(\nu_j)} \sum_{k=-\min(\nu_j)}^{n-\max(\nu_j)} \prod_{t \in \nu_j} X_{t+k}.$$

The previously discussed second- and third-order cumulant functions, as given by $s=2$ and $s=3$ in (8), simplify to the following centered expectations:

$$\begin{aligned} C(\tau_1) &= E[(X_t - \mu)(X_{t+\tau_1} - \mu)] \\ C(\tau_1, \tau_2) &= E[(X_t - \mu)(X_{t+\tau_1} - \mu)(X_{t+\tau_2} - \mu)]. \end{aligned}$$

In these cases, the corresponding estimator in (10) simplifies to

$$\widehat{C}(\boldsymbol{\tau}) = \frac{1}{n} \sum_{t=1}^{n-\gamma} \prod_{j=1}^s (X_{t-\alpha+\tau_j} - \bar{X}), \quad (11)$$

where $\alpha = \min(0, \tau_1, \dots, \tau_{s-1})$ and $\gamma = \max(0, \tau_1, \dots, \tau_{s-1}) - \alpha$, and \bar{X} represents the sample mean of the data. We extend the domain of \widehat{C} to all of \mathbb{Z}^s by defining $\widehat{C}(\boldsymbol{\tau}) = 0$ when the sum in (10) or (11) is empty.

Consistent estimation of the polyspectra (9) is obtained by taking the Fourier transform of the sample cumulant function, $\widehat{C}(\boldsymbol{\tau})$, multiplied by a smoothing kernel κ_m with bandwidth $m = m(n)$ that grows asymptotically with n but with $m/n \rightarrow 0$; in other words, let

$$\hat{f}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^{s-1}} \sum_{\|\boldsymbol{\tau}\| < n} \kappa_m(\boldsymbol{\tau}) \widehat{C}(\boldsymbol{\tau}) e^{-i\boldsymbol{\tau} \cdot \boldsymbol{\omega}}. \quad (12)$$

Typically, the kernel κ_m is obtained by ‘‘dilation’’ of a fixed underlying kernel κ , i.e., letting $\kappa_m(\boldsymbol{\tau}) = \kappa(\boldsymbol{\tau}/m)$. Several different shapes for κ have been proposed in the

literature, particularly for second-order spectral density estimation; cf. the study by Priestley (1983). In particular, utilizing a “flat-top” lag-window function, such as the trapezoidal function (Politis and Romano, 1995) or the conical frustum (Politis, 2011), will yield a (poly)spectral density estimate with optimal mean square error properties.

Asymptotic theory of the kernel-based polyspectral density estimators (12) is detailed in the works done by Berg and Politis (2009), Brillinger and Rosenblatt (1967), and Rosenblatt (1985). Two assumptions are generally required:

ASSUMPTION 1. *The cumulant function $C(\tau_1, \dots, \tau_{s-1})$ satisfies*

$$\sum_{(t_1, \dots, t_{s-1}) \in \mathbb{Z}^{s-1}} t_j C(t_1, \dots, t_{s-1}) \quad \text{for each } j = 1, \dots, s-1.$$

This assumption implies the existence of a continuously differentiable polyspectral density.

ASSUMPTION 2. *The kernel $\kappa(\boldsymbol{\tau})$ is continuously differentiable and satisfies*

$$\max \left(|\tau_j \kappa(\boldsymbol{\tau})|, \left| \frac{\partial}{\partial \tau_j} \kappa(\boldsymbol{\tau}) \right| \right) \leq M(1 + \|\boldsymbol{\tau}\|)^{-(s-1)-\epsilon} \quad \text{for each } j = 1, \dots, s-1,$$

where $\|\boldsymbol{\tau}\| = \left(\sum_{j=1}^{s-1} \tau_j^2 \right)^{1/2}$, $M > 0$, and $\epsilon > 0$.

If $\{X_t\}$ is a strictly stationary process, Assumptions 1 and 2 can be used to show that

$$\sqrt{n/m^{s-1}} \left(\hat{f}(\boldsymbol{\omega}) - \mathbb{E} \left[\hat{f}(\boldsymbol{\omega}) \right] \right) \longrightarrow_d \mathcal{N}(0, \sigma^2) \quad (13)$$

when $n \rightarrow \infty$ but $n/m^{s-1} \rightarrow \infty$; here σ^2 is a complex-valued functional of f and κ .

REMARK 1. If the bias of $\hat{f}(\boldsymbol{\omega})$ is of smaller order than $\sqrt{n/m^{s-1}}$, then $\mathbb{E}[\hat{f}(\boldsymbol{\omega})]$ in (13) can be replaced with $f(\boldsymbol{\omega})$. This minimal bias property can be achieved in two ways: (1) by selecting a bandwidth m that is (slightly) bigger than the optimal one resulting in a certain *undersmoothing*, or (2) by using an infinite-order kernel κ , which possesses reduced bias properties (Politis, 2011). Selecting an optimal bandwidth in finite samples is an unavoidable issue in nonparametric function estimation; a practical and effective method for selecting an appropriate bandwidth for polyspectral estimation is given in the study by Berg and Politis (2009). \square

5.2. The test statistics t_n^G and t_n^L

Due to the symmetries inherent to polyspectra (Berg, 2008), the normalized bispectrum, $K(\omega_1, \omega_2)$ is uniquely defined by its values on Ω given by

$$\Omega := \{(\omega_1, \omega_2): 0 < \omega_1 < \pi, 0 < \omega_2 < \min(\omega_1, 2(\pi - \omega_1))\}.$$

Utilizing estimates of the polyspectra in (12) yields $\hat{K}(\omega_1, \omega_2)$, the estimator of the normalized bispectrum. The Subba Rao and Gabr (1984) Gaussianity test statistic is then defined as

$$t_n^G = \sum_{j=1}^k \hat{K}(\omega_j^1, \omega_j^2), \quad (14)$$

where (ω_j^1, ω_j^2) ($j = 1, \dots, k$) constitutes a grid of k points inside Ω ; the number of points k increases with n to ensure consistency of the test. The null hypothesis of Gaussianity is rejected if t_n^G is too large.

Hinich (1982) proposed an improved and more robust version of the original bispectrum-based linearity test proposed by Subba Rao and Gabr. The Hinich linearity test statistic is given as

$$t_n^L = IQR \left\{ \left[\hat{K}(\omega_j^1, \omega_j^2) \right]_{j=1}^k \right\}, \quad (15)$$

where IQR stands for the interquartile range. The null hypothesis of linearity is rejected if t_n^L is too large.

In either case, t_n^G or t_n^L , the practitioner must determine the threshold of the critical region, i.e., decide what constitutes “too large” a value of the test statistic. This has been traditionally accomplished via asymptotic arguments (Hinich, 1982, Subba Rao and Gabr, 1984). However, as discussed in Section 4, we can alternatively determine the threshold by a resampling approximation offered by the AR-sieve bootstrap. The following Section describes how to obtain a subsampling approximation to such a critical region.

5.3. Subsampling for t_n^G and t_n^L

In order to establish the consistency of subsampling for the test statistics t_n^G and t_n^L , it must be shown that their sampling distribution converges to a continuous limit law under their respective null hypothesis. The asymptotics of the t_n^G and t_n^L have been established in the literature as presented below.

If the time series is Gaussian, then (Subba Rao and Gabr, 1980, 1984)

$$\left(\frac{n}{m^2} \cdot \frac{2\pi}{\zeta_2} \right) t_n^G \longrightarrow_d \chi_{2k}^2, \quad (16)$$

where $m = m(n)$ is the bandwidth used for the estimator (12) and $\zeta_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \kappa^2(\tau_1, \tau_2) d\tau_1 d\tau_2$.

If the time series is linear, then (Berg et al., 2010, Hinich, 1982)

$$\left(\frac{n}{m^2} \cdot \frac{2\pi}{\zeta_2} \right) t_n^L \longrightarrow_d N \left[\xi_{3/4} - \xi_{1/4}, \frac{1}{16k} \left(\frac{3}{g^2(\xi_{1/4})} + \frac{3}{g^2(\xi_{3/4})} - \frac{2}{g^2(\xi_{1/4})g^2(\xi_{3/4})} \right) \right], \quad (17)$$

where ξ . and $g(\cdot)$ are the quantile and density functions, respectively, of the χ_{2k}^2 distribution.

Let t_n denote either t_n^G or t_n^L as appropriate. It is easy to see that t_n satisfies the property $t_n \rightarrow 0$ under its respective null and $t_n \rightarrow t > 0$ under the alternative; convergence of t_n under the alternative is investigated in the works done by [Hinich \(1982\)](#). Define $t_{n,b,t}$ to be the statistic defined by (14) or (15), whichever appropriate, calculated using only the subsample $\{X_t, X_{t+1}, \dots, X_{t+b-1}\}$ for $t \in \{1, 2, \dots, n - b + 1\}$.

We now consider two candidate subsampling distributions for subsampling the hypothesis test of Gaussianity or linearity. First we define the uncentered subsampling distribution as presented in the study by [Politis et al. \(1999\)](#),

$$S_{n,b}^U(x) := \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} 1\{\tau_b t_{n,b,t} \leq x\}, \quad (18)$$

where⁴ $\tau_b = b/m(b)^2$.

Alternatively, a centered version of the above subsampling distribution has been shown to possess improved power in many contexts ([Berg et al., 2010](#)). The centered subsampling distribution is given by

$$S_{n,b}^C(x) := \frac{1}{n - b + 1} \sum_{t=1}^{n-b+1} 1\{\tau_b(t_{n,b,t} - t_n) \leq x\}. \quad (19)$$

It follows from (16) and (17) that the sampling distribution of $\tau_n t_n$ converges, under the respective null hypothesis, to a continuous limit law with cumulative distribution function denoted by $H(x)$. The consistency of the subsampling method as applied to linearity and Gaussianity testing is now stated; the following theorem follows directly from Theorem 3.5.1 in [Politis et al. \(1999\)](#).

THEOREM 1 (Validity of subsampling for t_n^G and t_n^L). *Let $H_{n,b}(x)$ denote either $S_{n,b}^U(x)$ or $S_{n,b}^C(x)$. Assume either (16) or (17) according to whether T_n denotes t_n^G or t_n^L . Assume $b \rightarrow \infty$, $b/n \rightarrow 0$ and $\tau_b/\tau_n \rightarrow 0$ as $n \rightarrow \infty$. Assume the bandwidth m for the polyspectra estimates used in the construction of the test statistics obeys the under-smoothing condition outlined in [Remark 1](#). Further assume the time series $\{X_t\}$ is strictly stationary, and strong mixing. For $\alpha \in (0, 1)$, define the two quantities*

$$\begin{aligned} h_{n,b}(1 - \alpha) &= \inf\{x : H_{n,b}(x) \geq 1 - \alpha\} \\ h(1 - \alpha) &= \inf\{x : H(x) \geq 1 - \alpha\} \end{aligned}$$

Then under the null hypothesis

- i. $h_{n,b}(1 - \alpha) \rightarrow g(1 - \alpha)$ in probability;
- ii. $\text{Prob}\{\tau_n t_n > h_{n,b}(1 - \alpha)\} \rightarrow \alpha$ as $n \rightarrow \infty$.

⁴ Recall that $m = m(n)$; for example, if $m(n) = n^\delta$ for some $\delta \in (0, 1/2)$, then $\tau_b = b/[b^\delta]^2 = b^{1-2\delta}$.

And under the alternative hypothesis,

$$\text{iii. } \text{Prob}\{\tau_n t_n > h_{n,b}(1 - \alpha)\} \longrightarrow 1 \quad \text{as } n \rightarrow \infty.$$

The above theorem shows that both subsampling distributions $S_{n,b}^U(x)$ or $S_{n,b}^C(x)$ yield consistent α -level tests. However, by analogy to other simpler examples (Berg et al., 2010), we expect that the test based on the centered subsampling distribution $S_{n,b}^C(x)$ would be more powerful than the one based on $S_{n,b}^U(x)$, i.e., that the convergence in part (iii) of the Theorem would be faster when $H_{n,b}(x) = S_{n,b}^C(x)$. By the same token, the convergence in part (ii) of the Theorem is expected to be faster when $H_{n,b}(x) = S_{n,b}^U(x)$, i.e., the level of the test would be more accurately achieved with the uncentered subsampling distribution $S_{n,b}^U(x)$.

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